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Hořava-Lifshitz $f(R)$ gravity

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Abstract: This paper is devoted to the construction of new type of $f(R)$ theories of gravity that are based on the principle of detailed balance. We discuss two versions of these theories with and without the projectability condition.

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1 Introduction

Recently in series of very interesting papers P. Hořava suggested new approach for the study of membranes and quantum gravity theories known as Hořava-Lifshitz gravities [1–4].[1] The attractive property of Hořava-Lifshitz gravity is that it is power-counting renormalizable. The second important property of the Hořava construction is *detailed balance condition*. This condition is based on the idea that the potential term in the Lagrangian of $D+1$ dimensional theory descants from the variation of D dimensional action. In fact, this construction is based on the following idea known from the condensed matter physics [58]: *Is it possible to find such a $D+1$ dimensional quantum theory such that its ground state wave functional reproduces the partition function of D dimensional theory?* This idea was elaborated in details in [58] and recently in series of papers by P. Hořava with many interesting results. In particular, if we start with known classical universality classes in D dimension we can construct a quantum critical systems in $D+1$ dimensions.

It is very interesting that similar situation naturally occurs in case of topological string theory [59, 60], OSV conjecture, topological M-theory, together with non-critical M-theory [61–63].[2]

As was carefully discussed recently in [11], there are at least four versions of the theory: with/without the detailed balance condition; and with/without the projectability condition. As we will show bellow the projectability condition means that the lapse function depends on time only. It was argued in many papers that the most promising is the version

[1]Hořava's ideas were elaborated from different points of view in couple of papers, see for example [5–57, 68–72].

[2]For recent discussion and extensive list of references, see [64].

without the detailed balance condition with the projectability condition that has a potential to be theoretically consistent and cosmologically viable.

Even if there are doubts considering the detailed balance condition in general relativity we feel that it deserves to be studied further. In particular, let us consider following situation when we have a D dimensional quantum theory with corresponding partition function $\mathcal{Z}$. Then we consider Hamiltonian of $D+1$ dimensional theory and ask the question under which condition this Hamiltonian annihilates the vacuum wave functional of given $D+1$ dimensional theory where the norm of this vacuum state coincides with the partition function $\mathcal{Z}$. We show that there exists an infinite number of such Hamiltonians that can be defined as a Taylor series in powers of creation and annihilation operators where the annihilation operator annihilates the vacuum wave functional.

We apply this construction to the case of theory of gravity and consider two situations. In the first one we follow the original Hořava's approach [3] where we start with partition function of D dimensional gravity and demand that there exists quantum gravity in $D+1$ dimension such that the norm of the ground state functional coincides with the partition function of D dimensional theory. Then we show that we can construct infinite number of Hamiltonians that annihilate this ground state. In other words we find an infinite number of Hamiltonians that obey the detailed balance conditions. Clearly these Hamiltonians are well defined at the classical level due to the well known peculiarities that arise in non-linear quantum theories. Even at the classical level these Hamiltonians have many interesting properties that should be studied further. For example, for the special form of the Hamiltonian that will be specified below we determine corresponding Lagrangian and we find that the action for this theory is manifestly invariant under spatial diffeomorphism. Then, following [3] we perform an extension of given symmetries that leads to the action that is invariant under *foliation preserving diffeomorphism*. We find new non-linear theory of gravity that resemble $f(R)$ theory of gravity[3] that however is not invariant under full $D+$ 1 diffeomorphism. It would be certainly interesting to study the cosmological implications of this model exactly in the same way as in case of ordinary $f(R)$ theory of gravity. Certainly we can also consider the more general form of Hamiltonians then the Hamiltonian explicitly studied in this paper.

In the second case we consider an alternative form of the principle of detailed balance. We consider the situation when the Hamiltonian density of $D+1$ dimensional theory is a linear combination of the diffeomorphism and Hamiltonian constraint. Further we assume that the Hamiltonian constraint has the special property that it annihilates the vacuum wave functional that has the norm equal to the partition function of D dimensional theory of gravity. Since this vacuum wave functional is manifestly invariant under D dimensional spatial diffeomorphisms it is annihilated by the generator of diffeomorphism and consequently by the Hamiltonian of the theory. We would like to stress that at this moment we only assume that the Hamiltonian annihilates the vacuum wave functional but we do not demand that it should annihilate all states of the theory. On the other hand we will argue that the Hamiltonian framework implies that the Hamiltonian should annihilate all states

[3]For review and extensive list of references, see [65, 73, 74].

in case of the quantum mechanical formulation of the theory. Explicitly, following the standard approach we determine an action corresponding to given Hamiltonian. Then we continue in the study of this theory and develop the Hamiltonian formalism that follows from this action. Since the action contains the fields $N(t, \mathbf{x})$ and $N_i(t, \mathbf{x})$ without time derivatives we find that the absence of corresponding momenta imply the primary constraints of the theory $\pi_N(\mathbf{x}) \approx 0\,, \pi^i(\mathbf{x}) \approx 0$. The consistency of these constraints with the time evolution of the system implies the secondary constraints $\mathcal{H}_0(\mathbf{x}) \approx 0\,, \quad \mathcal{H}^i(\mathbf{x}) \approx 0$. Following the Dirac approach we then find that *all* quantum states of the theory have to be annihilated by these constraints as opposite to the original assumption that the state that should be annihilated by H is the vacuum state only. On the other hand we will argue that these theories suffer from the same problems as the Hořava's theories without projectability conditions [37]. However using the fact that these theories are constructed as theories that obey the detailed balance conditions it is possible to find the algebra of constraints that close however that do not support any physical excitations at all. In other words these theories are topological. As the second example of solvable theory we consider the case of ultralocal theory and we argue the algebra of constraints closes as in standard gravity theory.

Let us outline our results. Imposing the detailed balance condition we are able to find new $D + 1$ $f(R)$ theories of gravity with or without projectability conditions. We would like to stress that these theories should be considered as toy models of gravity theories. It would be interesting to study the cosmological implications of these theories in the same way as in case of $f(R)$ theories of gravity with full diffeomorphism invariance.

This paper is organized as follows. In next section (2) we present the main idea of our construction on the simple case of collection of D scalar fields in $p + 1$ dimensions. Then in section (3) we perform the construction of new $D + 1$ dimensional theory of gravity that is invariant under foliation preserving diffeomorphism. In section (4) we suggest an alternative way how to impose the condition of the detailed balance in case of $D + 1$ dimensional theory of gravity and we argue that this procedure leads to $D+1$ dimensional theory without projectability condition.

2 Non-linear scalar Lifshitz theory

In this section we describe the construction of non-linear Lifshitz theory based on the detailed balance condition on the simple example of collection of D scalar fields in p dimensions. This procedure is based on an idea is that the norm of the ground state functional of $p + 1$ dimensional theory coincides with the partition function of any p dimensional theory. We should stress that this requirement is pure formal since we do not carry about issues whether this partition function is well defined. Very nice discussion of issues that are related to the construction of wave functionals can be found in paper [66]. Despite of this fact we proceed further and we find that we are able to find new interesting class of theories at least at the classical level.

Let us start in the same way as in [3, 58] and consider the situation when we have a collection of D scalar fields defined on p dimensional Euclidean space with coordinates

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$\mathbf{x} = (x^i)\,, i = 1, \ldots, p$ with following action

$$W = \frac{1}{2}\int d^p\mathbf{x}\delta^{ij}\partial_i\Phi^M\partial_j\Phi^N g_{MN}\,, \tag{2.1}$$

where $g_{MN}\,, M, N = 1, \ldots, D$ is a *constant positive definite symmetric* matrix. Clearly we can consider more general form of the action than the one given in (2.1) but in order to explain the main idea of the constructions we restrict ourselves to the simple action given above.

As in standard quantum mechanics the fundamental object of this theory is the partition function $\mathcal{Z}$

$$\mathcal{Z} = \int \mathcal{D}\Phi(\mathbf{x})\exp[-W(\Phi(\mathbf{x}))] \tag{2.2}$$

that is defined as a path integral on the space of field configurations $\Phi^M(\mathbf{x})$. Then let us assume an existence of $p+1$ dimensional quantum field theory with collection of the operators $\hat{\Phi}^M(\mathbf{x})$ and their conjugate momenta $\hat{\Pi}_M(\mathbf{x})$ and that obey the canonical commutation relation

$$[\hat{\Phi}^M(\mathbf{x}), \hat{\Pi}_N(\mathbf{y})] = i\delta^M_N\delta(\mathbf{x}-\mathbf{y})\ . \tag{2.3}$$

Further, we introduce eigenstate of $\hat{\Phi}^M(\mathbf{x})$ that is the state $|\Phi(\mathbf{x})\rangle$ that obeys

$$\hat{\Phi}^M(\mathbf{x})\,|\Phi(\mathbf{x})\rangle = \Phi^M(\mathbf{x})\,|\Phi(\mathbf{x})\rangle\ . \tag{2.4}$$

In the Schrödinger representation any state of given system is represented as the state functional $\Psi[\Phi(\mathbf{x})]$ and the standard interpretation of quantum mechanics implies that $\Psi[\Phi(\mathbf{x})]\Psi^*[\Phi(\mathbf{x})]$ is a density on the configuration space. Note also that action of the operator $\hat{\Phi}^M(\mathbf{x})$ on this state functional corresponds to multiplication with $\Phi^M(\mathbf{x})$. On the other hand the commutation relation (2.3) implies that in the Schrödinger representation the operator $\hat{\Pi}_M(\mathbf{x})$ is equal to

$$\hat{\Pi}_M(\mathbf{x}) = -i\frac{\delta}{\delta\Phi^M(\mathbf{x})}\ . \tag{2.5}$$

Our goal is to formulate $p+1$ dimensional system with the property that the norm of its ground-state functional $\Psi_0[\Phi(\mathbf{x})]$ reproduces the partition function (2.2)

$$\langle\Psi_0|\Psi_0\rangle = \int\mathcal{D}\Phi(\mathbf{x})\Psi^*_0[\Phi(\mathbf{x})]\Psi_0[\Phi(\mathbf{x})] = \int\mathcal{D}\Phi(\mathbf{x})\exp[-W(\Phi(\mathbf{x}))]\ . \tag{2.6}$$

Everything that has been done up to this point is well known. However we now make a presumption that the Hamiltonian of $p+1$ dimensional theory has the form

$$\hat{\mathcal{H}}(\mathbf{x}) = \kappa^2\left(\sum_{n=0}^{\infty}\hat{c}_n(\hat{\Phi})(\hat{Q}^\dagger_M g^{MN}\hat{Q}_N)^n - \hat{c}_0(\hat{\Phi})\right)\,,$$

where κ is a coupling constant, $\hat{c}_n(\hat{\Phi})$ are functions that generally depend on the operators $\hat{\Phi}$ and where $\hat{Q}_M, \hat{Q}^\dagger_M$ are defined as

$$\hat{Q}_M = i\hat{\Pi}_M + \frac{1}{2}\frac{\delta W[\hat{\Phi}]}{\delta\hat{\Phi}^M(\mathbf{x})}\,,\quad \hat{Q}^\dagger_M = -i\hat{\Pi}_M + \frac{1}{2}\frac{\delta W[\hat{\Phi}]}{\delta\hat{\Phi}^M(\mathbf{x})}\ . \tag{2.7}$$

Note that in the Schrödinger representation the operators $\hat{Q}_M, \hat{Q}^\dagger_M$ are equal to

$$\hat{Q}_M = \frac{\delta}{\delta \Phi^M(\mathbf{x})} + \frac{1}{2}\frac{\delta W[\Phi]}{\delta \Phi^M(\mathbf{x})} , \quad \hat{Q}^\dagger_M = -\frac{\delta}{\delta \Phi^M(\mathbf{x})} + \frac{1}{2}\frac{\delta W[\Phi]}{\delta \Phi^M(\mathbf{x})} . \tag{2.8}$$

Let us assume that the vacuum wave functional takes the form

$$\Psi_0[\Phi(\mathbf{x})] = \exp\left(-\frac{1}{2}W\right) = \exp\left(-\frac{1}{4}\int d^p\mathbf{x}\delta^{ij}\partial_i\Phi^M(\mathbf{x})g_{MN}\partial_j\Phi^N(\mathbf{x})\right) . \tag{2.9}$$

Then it is easy that $\hat{Q}_M$ defined in (2.8) annihilates Ψ_0

$$\hat{Q}_M\Psi[\Phi(\mathbf{x})] = 0 \tag{2.10}$$

as follows from the fact that

$$i\hat{\Pi}(\mathbf{x})\Psi_0[\Phi] = \frac{\delta}{\delta\Phi^M(\mathbf{x})}\Psi_0[\Phi] = -\frac{1}{2}\frac{\delta W}{\delta \Phi^M(\mathbf{x})}\Psi_0[\Phi] . \tag{2.11}$$

In other words the vacuum wave functional is annihilated by $\hat{Q}_M$ and by construction it is an eigenstate of the Hamiltonian with zero energy. Further, the norm of the vacuum wave functional coincides with the partition function of p dimensional theory.

It is clear that it this way we can define an infinite number of Hamiltonians that obey the detailed balance condition. In what follows we restrict ourselves to the following form of the Hamiltonian density[4]

$$\hat{\mathcal{H}} = \kappa^2\left(\sqrt{\hat{\alpha}(\hat{\Phi}) + \hat{\beta}(\hat{\Phi})\left(\hat{\Pi}_M g^{MN}\hat{\Pi}_N + \frac{1}{4}\left(\frac{\delta W}{\delta\hat{\Phi}^M}g_{MN}\frac{\delta W}{\delta\hat{\Phi}^N}\right)\right)} - \sqrt{\hat{\alpha}(\hat{\Phi})}\right) , \tag{2.12}$$

where $\hat{\alpha}, \hat{\beta}$ generally depend on $\hat{\Phi}$ and where the square root function in the definition of the Hamiltonian is defined as the Taylor polynomial in $(\hat{Q}^\dagger\hat{Q})^n$ written explicitly in (2.7).

As the next step we determine the Lagrangian from the classical form of the Hamiltonian density (2.12). Using the Hamiltonian equation $\partial_t\Phi = \{\Phi, H\}$ and the form of the Hamiltonian density (2.12) we find

$$\partial_t\Phi^M = \{\Phi^M, H\} = \kappa^2\frac{\beta\Pi_N g^{NM}}{\sqrt{\alpha + \beta(\Pi_M g^{MN}\Pi_N + \frac{1}{4}\frac{\delta W}{\delta\Phi^M}g^{MN}\frac{\delta W}{\delta\Phi^N})}} \tag{2.13}$$

so that the Lagrangian density is equal to

$$\begin{aligned}\mathcal{L} &= \Pi_M\partial_t\Phi^M - \mathcal{H} = \\ &= -\kappa^2\sqrt{\alpha(\Phi) + \frac{\beta(\Phi)}{4}\frac{\delta W}{\delta\Phi^M}g^{MN}\frac{\delta W}{\delta\Phi^N}}\sqrt{1 - \frac{1}{\kappa^4\beta(\Phi)}\partial_t\Phi^M g_{MN}\partial_t\Phi^N} + \kappa^2\sqrt{\alpha(\Phi)}.\end{aligned} \tag{2.14}$$

[4]The reason for this choose of the Hamiltonian density is that its functional form allows to easily find the closed relation between time derivatives of canonical variables and conjugate momenta. Clearly this is not the case for general form of the Hamiltonian density (2.12). The analysis of more general forms of Hamiltonian densities will be given in future.

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Let us now simplify the action further and consider the case when $\alpha = 1, \beta = \text{const}$. Then, since the variation of (2.1) is equal to

$$\frac{\delta W}{\delta \Phi^M(\mathbf{x})} = -\partial^i \partial_i \Phi^N(\mathbf{x}) g_{NM} \tag{2.15}$$

we find that the action of $p+1$ dimensional theory takes the form

$$S = -\kappa^2 \int d^p \mathbf{x} dt \sqrt{1 + \frac{\beta}{4}(\partial^i \partial_i \Phi^M) g_{MN} (\partial^j \partial_j \Phi^N)} \sqrt{1 - \frac{1}{\kappa^4 \beta} \partial_t \Phi^M g_{MN} \partial_t \Phi^N} . \tag{2.16}$$

If we now expand this action up to quadratic order in fields we find the standard Lifshitz action (up to trivial rescaling of β)

$$S = -\kappa^2 \int dt d^p \mathbf{x} - \int dt d^p \mathbf{x} [\kappa^2 \beta \frac{1}{8} (\partial^i \partial_i \Phi^M) g_{MN} (\partial^j \partial_j \Phi^N) - \frac{1}{2\kappa^2 \beta} \partial_t \Phi^M g_{MN} \partial_t \Phi^N] . \tag{2.17}$$

In other words for small spatial and time derivatives the Lagrangian (2.14) reduces to the Lifshitz scalar theory.

3 Hořava-Lifshitz $f(R)$ theory of gravity-with projectability condition

Let us now turn to the main topic of this paper which is a construction of the Hořava-Lifshitz $f(R)$ theories of gravity in $D+1$ dimensions. This construction is based on assumption that we have $D+1$ dimensional quantum theory of gravity that is characterized by following quantum Hamiltonian density

$$\hat{\mathcal{H}} = \kappa^2 \sqrt{\hat{g}} \left(\sum_{n=0}^{\infty} \hat{c}_n(\hat{g}_{ij}) \left(\hat{Q}^{\dagger ij} \frac{1}{\hat{g}} \hat{\mathcal{G}}_{ijkl} \hat{Q}^{kl} \right)^n - \hat{c}_0(\hat{g}_{ij}) \right) , \tag{3.1}$$

where

$$\hat{Q}^{\dagger ij} = -i\hat{\pi}^{ij} + \sqrt{\hat{g}} \hat{E}^{ij}(\hat{g}_{ij}) , \quad \hat{Q}^{ij} = i\hat{\pi}^{ij} + \sqrt{\hat{g}} \hat{E}^{ij}(\hat{g}_{ij}) , \tag{3.2}$$

and where $\hat{g} = \det \hat{g}_{ij}$ and κ is a coupling constant of given theory. Note that the fundamental operators of quantum theory of gravity are metric components $\hat{g}_{ij}(\mathbf{x}) , i = 1, \dots, D$ together with their conjugate momenta $\hat{\pi}^{ij}(\mathbf{x})$. These operators obey the commutation relations

$$[\hat{g}_{ij}(\mathbf{x}), \hat{\pi}^{kl}(\mathbf{y})] = \frac{1}{2}(\delta_i^k \delta_j^l + \delta_i^l \delta_j^k) \delta(\mathbf{x} - \mathbf{y}) . \tag{3.3}$$

Further, $\hat{c}_n$ defined in (3.1) are scalar functions that depend on $\hat{g}_{ij}$ only. It is clear that in the Schrödinger representation the operators (3.2) take the form

$$\hat{Q}^{ij}(\mathbf{x}) = -\frac{\delta}{\delta g^{ij}(\mathbf{x})} + \sqrt{g}(\mathbf{x}) E^{ij}(\mathbf{x}) , \quad \hat{Q}^{\dagger ij}(\mathbf{x}) = \frac{\delta}{\delta g^{ij}(\mathbf{x})} + \sqrt{g}(\mathbf{x}) E^{ij}(\mathbf{x}) . \tag{3.4}$$

The next goal is to specify the form of the operators E^{ij}. To do this we assume that the theory obeys the *detailed balance condition* so that

$$\sqrt{g}(\mathbf{x}) E^{ij}(\mathbf{x}) = \frac{1}{2} \frac{\delta W}{\delta g^{ij}(\mathbf{x})} , \tag{3.5}$$

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where W is an action of D dimensional gravity. As in [3] we construct the vacuum wave functional of $D+1$ dimensional theory as

$$\Psi[g(\mathbf{x})] = \exp\left(-\frac{1}{2}W\right) , \tag{3.6}$$

where W is the Einstein-Hilbert action in D dimensions

$$W = \frac{1}{2\kappa_W^2}\int d^D\mathbf{x}\sqrt{g}R . \tag{3.7}$$

Generally the action W could also contains additional terms that are functions of metric however the explicit form of W will not be important in following discussion.

The form of the vacuum wave functional (3.6) implies that it is annihilated by (3.1). Further as a consequence of the detailed balance condition the norm of the functional (3.6) coincides with the partition function of D dimensional Euclidean gravity. In other words we have again infinite number of possible Hamiltonians that annihilate the vacuum state (3.6) and that are defined using the principle of detailed balance.

In order to find the Lagrangian formulation of this theory we now consider the classical form of the Hamiltonian density (3.1). In order to simplify the analysis we restrict ourselves to the following explicit form of the Hamiltonian density

$$\mathcal{H} = \kappa^2\sqrt{g}\left(\sqrt{1+\beta(-i\pi^{ij}+\sqrt{g}E^{ij})\frac{1}{g}\mathcal{G}_{ijkl}(i\pi^{kl}+\sqrt{g}E^{kl})}-1\right) , \tag{3.8}$$

where $\mathcal{G}_{ijkl}$ denotes the inverse of the De Witt metric

$$\mathcal{G}_{ijkl} = \frac{1}{2}(g_{ik}g_{jl}+g_{il}g_{jk})-\tilde{\lambda}g_{ij}g_{kl} \tag{3.9}$$

with $\tilde{\lambda} = \frac{\lambda}{D\lambda-1}$. The "metric on the space of metric", $\mathcal{G}^{ijkl}$ is defined as

$$\mathcal{G}^{ijkl} = \frac{1}{2}(g^{ik}g^{jl}+g^{il}g^{jk}-\lambda g^{ij}g^{kl}) \tag{3.10}$$

with λ an arbitrary real constant. Note that (3.9) together with (3.10) obey the relation[5]

$$\mathcal{G}_{ijmn}\mathcal{G}^{mnkl} = \frac{1}{2}(\delta_i^k\delta_j^l+\delta_i^l\delta_j^k) . \tag{3.11}$$

The form of the Hamiltonian density (3.8) implies following time derivative of g_{ij}

$$\partial_t g_{ij} = \{g_{ij},H\} = \kappa^2\frac{\beta\mathcal{G}_{ijkl}\pi^{kl}}{\sqrt{g}\sqrt{1+\beta(-i\pi^{ij}+\sqrt{g}E^{ij})\frac{1}{g}\mathcal{G}_{ijkl}(i\pi^{kl}+\sqrt{g}E^{kl})}}. \tag{3.12}$$

[5]Note that we use the terminology introduced in [3] and that we review there. In case of relativistic theory, the full diffeomorphism invariance fixes the value of λ uniquely to equal $\lambda = 1$. In this case the object $\mathcal{G}_{ijkl}$ is known as the "De Witt metric". We use this terminology to more general case when λ is not necessarily equal to one.

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With the help of this result we can express π^{ij} as a function of g_{ij} and $\partial_t g_{ij}$. Then we easily find the corresponding Lagrangian density in the form

$$\mathcal{L} = \partial_t g_{ij}\pi^{ij} - \mathcal{H} = -\kappa^2\sqrt{g}\left(\sqrt{1+\beta E^{ij}\mathcal{G}_{ijkl}E^{kl}}\sqrt{1-\frac{1}{\kappa^4\beta}\partial_t g_{ij}\mathcal{G}^{ijkl}\partial_t g_{kl}}-1\right). \quad (3.13)$$

By construction the action

$$S = \int d^D\mathbf{x}\mathcal{L}\,, \quad (3.14)$$

where $\mathcal{L}$ is given in (3.13) is invariant under the global time translation $t' = t+\delta t\,, \delta t = \mathrm{const}$ and under the spatial diffeomorphism

$$x'^i = x^i(\mathbf{x})\ . \quad (3.15)$$

This follows from the fact that we presumed that the functional W is invariant under the spatial diffeomorphism under which the metric g_{ij} and tensor E^{ij} transform as

$$\begin{aligned} g'_{ij}(\mathbf{x}') &= g_{kl}(\mathbf{x})\left(D^{-1}\right)^k_i\left(D^{-1}\right)^l_j\,, \\ E'^{ij}(\mathbf{x}') &= E^{kl}(\mathbf{x})D^i_k D^j_l\,, \end{aligned} \quad (3.16)$$

where

$$D^i_j = \frac{\partial x'^i}{\partial x^j}\,, \quad D^i_j\left(D^{-1}\right)^j_k = \delta^i_k\ . \quad (3.17)$$

Using the transformation property of g_{ij} we find that the metric $\mathcal{G}_{ijkl}$ transforms as

$$\mathcal{G}'_{ijkl}(\mathbf{x}') = \mathcal{G}_{i'j'k'l'}(\mathbf{x})\left(D^{-1}\right)^{i'}_i\left(D^{-1}\right)^{j'}_j\left(D^{-1}\right)^{k'}_k\left(D^{-1}\right)^{l'}_l \quad (3.18)$$

and the invariance of the action under the spatial diffeomorphism (3.15) is obvious.

3.1 Extension of symmetries

We argued that the action formulated above is invariant under D dimensional *spatial diffeomorphism*. As in [2, 3] we extend these symmetries to the diffeomorphisms that respect the preferred codimension-one foliation $\mathcal{F}$ of the theory by the slices of fixed time. By definition such a foliation-preserving diffeomorphism consists a space-time dependent spatial diffeomorphisms as well as time-dependent time reparameterization. These symmetries are now generated by infinitesimal transformations

$$\delta x^i \equiv x'^i - x^i = \zeta^i(t,\mathbf{x})\,, \quad \delta t \equiv t'-t = f(t)\ . \quad (3.19)$$

It was shown in [3] that the metric transform under (3.19) as

$$g'_{ij}(t',\mathbf{x}') = g_{ij}(t,\mathbf{x}) - g_{il}(t,\mathbf{x})\partial_j\zeta^l(t,\mathbf{x}) - \partial_i\zeta^k(t,\mathbf{x})g_{kj}(t,\mathbf{x}). \quad (3.20)$$

The original action (3.14) is not invariant under (3.19). On the other hand it was shown in [3] that in order to find an action that is invariant under (3.19) it is necessary to introduce new fields $N_i(t,\mathbf{x}), N(t)$ that transform under (3.19) as

$$\begin{aligned} N'_i(t',\mathbf{x}') &= N_i(t,\mathbf{x}) - N_i(t,\mathbf{x})\dot{f}(t) - N_j(t,\mathbf{x})\partial_i\zeta^j(t,\mathbf{x}) - g_{ij}(t,\mathbf{x})\dot{\zeta}^j(t,\mathbf{x})\,, \\ N'(t') &= N(t) - N(t)\dot{f}(t). \end{aligned} \quad (3.21)$$

As the next step we have to replace volume element $dtd^D\mathbf{x}\sqrt{g}$ with $dtd^D\mathbf{x}N\sqrt{g}$ and the time derivative of g_{ij} with

$$\partial_t g_{ij} \Rightarrow 2K_{ij} \ , \tag{3.22}$$

where K_{ij} is defined as

$$K_{ij} = \frac{1}{2N}(\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i) \, , \tag{3.23}$$

and where ∇_i is D dimensional covariant derivative constructed from the metric components g_{ij}. It can be shown that (3.23) transform covariantly under (3.19)

$$K'_{ij}(t', \mathbf{x}') = K_{ij}(t, \mathbf{x}) - K_{ik}(t, \mathbf{x})\partial_j \zeta^k(t, \mathbf{x}) - \partial_i \zeta^k(t, \mathbf{x}) K_{kj}(t, \mathbf{x}). \tag{3.24}$$

Performing these substitutions in (3.14) we find the gravity action invariant under the foliation preserving diffeomorphism in the form

$$S = -\kappa^2 \int dtd^D\mathbf{x}\sqrt{g}N \left(\sqrt{1 + \beta E^{ij}\mathcal{G}_{ijkl}E^{kl}} \sqrt{1 - \frac{4}{\kappa^4 \beta}(K_{ij}K^{ij} - \lambda K^2)} - 1 \right) . \tag{3.25}$$

Note also that linearized form of the action (3.25) takes the form

$$S = \frac{1}{2} \int dtd^D\mathbf{x}\sqrt{g}N \left(\frac{4}{\kappa^2 \beta}(K_{ij}K^{ij} - \lambda K^2) - \kappa^2 \beta E^{ij}\mathcal{G}_{ijkl}E^{kl} \right) \tag{3.26}$$

that after trivial rescaling of parameter β resembles the Hořava's form of the gravity theory. For that reason we can consider the action (3.25) as the $f(R)$-like version of the Horařava-Lifshitz gravity.

In the next subsection we develop the Hamiltonian formalism of given theory.

3.2 Hamiltonian formalism

The dynamical variables of the theory are $N_i(\mathbf{x}), \pi^i(\mathbf{x}), N, \pi^N$ together with $g_{ij}(\mathbf{x}), \pi^{ij}(\mathbf{x})$ with corresponding non-zero Poisson brackets

$$\left\{ g_{ij}(\mathbf{x}), \pi^{kl}(\mathbf{y}) \right\} = \frac{1}{2}(\delta_i^k \delta_j^l + \delta_i^l \delta_j^k)\delta(\mathbf{x} - \mathbf{y}) \ , \quad \left\{ N^i(\mathbf{x}), \pi_j(\mathbf{y}) \right\} = \delta_j^i \delta(\mathbf{x} - \mathbf{y}) \, , \quad \left\{ N, \pi^N \right\} = 1 \, . \tag{3.27}$$

Note that $N(t)$ and $\pi^N(t)$ are homogeneous functions of time. In other words they obey projectability condition which has an important consequence for the consistency of the Hořava-Lifshitz theory [11]. Further, as follows from the form of the action (3.25) the momenta π^{ij} conjugate to g_{ij} can be expressed as function of g_{ij} and $\partial_t g_{ij}$ from the relation

$$\pi^{ij}(\mathbf{x}) = \frac{\delta S}{\delta \partial_t g_{ij}(\mathbf{x})} = \frac{2}{\kappa^2 \beta} \frac{\sqrt{g}\mathcal{G}^{ijkl}K_{kl}}{\sqrt{1 - \frac{4}{\beta\kappa^4}K_{ij}\mathcal{G}^{ijkl}K_{kl}}} \sqrt{1 + \beta E^{ij}\mathcal{G}_{ijkl}E^{kl}} . \tag{3.28}$$

On the other hand since the time derivative of N_i and N do not appear in the action (3.25) we find that the momenta π^i and π^N form the primary constraints of the theory

$$\pi^i(t, \mathbf{x}) \approx 0 \, , \quad \pi^N(t) \approx 0 \ . \tag{3.29}$$

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Finally the standard definition of the Hamiltonian density gives

$$\begin{aligned}\mathcal{H} &= \partial_t g_{ij}\pi^{ij} - \mathcal{L} = \\ &= \kappa^2\sqrt{g}N\left(\sqrt{1+\frac{1}{g}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl}+\beta E^{ij}\mathcal{G}_{ijkl}E^{kl}}-1\right)+ \\ &\quad +(\nabla_i N_j+\nabla_j N_i)\pi^{ij}. \end{aligned} \tag{3.30}$$

As a consequence we find that the Hamiltonian is equal to

$$\begin{aligned} H &= \int d^D\mathbf{x}\mathcal{H} = \int d^D\mathbf{x}(N(t)\mathcal{H}_0(\mathbf{x},t)+N_i(t,\mathbf{x})\mathcal{H}^i(\mathbf{x},t))\,, \\ \mathcal{H}_0 &= \kappa^2\sqrt{g}\left(\sqrt{1+\frac{1}{g}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl}+E^{ij}\mathcal{G}_{ijkl}E^{kl}}-1\right)\,, \quad \mathcal{H}^i = -2\nabla_j\pi^{ij}\,, \end{aligned} \tag{3.31}$$

where we ignore boundary terms.

The primary constraints $\pi^i(\mathbf{x})\approx 0\,,\pi^N(t)\approx 0$ have to be preserved during the time evolution of the system and consequently

$$\partial_t\pi^i(\mathbf{x}) = \left\{\pi^i(\mathbf{x}),H\right\} = -\mathcal{H}^i(\mathbf{x})\approx 0\,, \quad \partial_t\pi^N(t) = \{N(t),H\} = -\int d^D\mathbf{x}\mathcal{H}_0(\mathbf{x})\approx 0\ . \tag{3.32}$$

Since the right side of the equations above have to vanish on constraint surface we find that the consistency of the primary constraints generate the secondary ones

$$\mathcal{H}^i(\mathbf{x})\approx 0\,, \quad \mathbf{T}_T\equiv\int d^D\mathbf{x}\mathcal{H}_0(\mathbf{x})\approx 0\ . \tag{3.33}$$

It is convenient to introduce the smeared form of the diffeomorphism constant $\mathbf{T}_S$ defined as

$$\mathbf{T}_S(\zeta) = \int d^D\mathbf{x}\zeta_i(\mathbf{x})\mathcal{H}^i(\mathbf{x}). \tag{3.34}$$

The next goal is to calculate the Poisson bracket of constraints $\mathbf{T}_T$ and $\mathbf{T}_S$. Trivially we have that

$$\{\mathbf{T}_T,\mathbf{T}_T\} = 0\ . \tag{3.35}$$

Now we calculate the Poisson brackets between $\mathbf{T}_S(\zeta)$ and $\mathbf{T}$

$$\{\mathbf{T}_S(\zeta),\mathbf{T}_T\} = -\int d^D\mathbf{x}(\zeta^k\partial_k\mathcal{H}_0-\partial_k(\mathcal{H}_0)\zeta^k) = 0\,, \tag{3.36}$$

where we used the Poisson bracket between $\mathbf{T}_S(\zeta)$ and $\mathcal{H}_0$[6]

$$\{\mathbf{T}_S(\zeta),\mathcal{H}_0\} = -\partial_k\zeta^k\mathcal{H}_0-\zeta^k\partial_k\mathcal{H}_0. \tag{3.37}$$

Finally we calculate the Poisson bracket

$$\{\mathbf{T}_S(\zeta),\mathbf{T}_S(\xi)\} = \mathbf{T}_S(\zeta^i\partial_i\xi-\xi^i\partial_i\zeta). \tag{3.38}$$

[6]For more detailed calculation, see (4.16).

In summary we find that the algebra of constraints for generalized Hořava-Lifshitz theory that respects the projectability condition takes very simple form

$$\begin{aligned}
\{\mathbf{T}_T, \mathbf{T}_T\} &= 0\,, \\
\{\mathbf{T}_S(\zeta), \mathbf{T}_T\} &= 0\,, \\
\{\mathbf{T}_S(\zeta), \mathbf{T}_S(\xi)\} &= \mathbf{T}_S(\zeta^i\partial_i\xi - \xi^i\partial_i\zeta).
\end{aligned} \tag{3.39}$$

The fact that the algebra of constraints is closed for any theory of gravity that obeys the projectability condition is very attractive. This result in contrast with the situation of gravity without the projectability condition when the algebra is not closed and the structure constants of the theory depend on phase space variables. On the other hand there are still many unsolved problems and issues considering Hořava-Lifshitz gravity theories as was reviewed carefully in [11, 16] so that these results should be taken with great care.

4 Hořava-Lifshitz $f(R)$ theory of gravity-without projectability condition

In this section we address the question of the formulation of the local form of the *condition of detailed balance*. We again start with the assumption that one can define the vacuum wave functional of $D+1$-dimensional quantum theory and that this functional has the form as in (3.6). Now we demand that this vacuum wave functional is annihilated by

$$\hat{H} = \int d^D\mathbf{x} \left(N(t,\mathbf{x})\hat{\mathcal{H}}_0(t,\mathbf{x}) + N_i(t,\mathbf{x})\hat{\mathcal{H}}^i(t,\mathbf{x}) \right)\,, \tag{4.1}$$

where $\hat{\mathcal{H}}^i$ is the generator of spatial diffeomorphism

$$\hat{\mathcal{H}}^i(\mathbf{x}) = -2\hat{\nabla}_j\hat{\pi}^{ji}(\mathbf{x})\,, \tag{4.2}$$

and where we assume that $\hat{\mathcal{H}}_0$ can be written as

$$\hat{\mathcal{H}}_0(\mathbf{x}) = \kappa^2\sqrt{\hat{g}}\left(\sum_{n=0}^{\infty}\hat{c}_n(\hat{g}_{ij})\left(\hat{Q}^{\dagger ij}\frac{1}{\hat{g}}\hat{\mathcal{G}}_{ijkl}\hat{Q}^{kl}\right)^n - \hat{c}_0(\hat{g}_{ij})\right)\,, \tag{4.3}$$

where $\hat{Q}^{ij}, \hat{Q}^{\dagger ij}$ were defined in (3.2) and the functional form of $\hat{E}^{ij}$ follows from (3.5). Then it is obvious that the local constraint $\hat{\mathcal{H}}_0(\mathbf{x})$ annihilates the vacuum wave functional (3.6). Since W is invariant under spatial diffeomorphism by construction we find that the vacuum wave functional Ψ_0 is annihilated by $\hat{H}$ (4.1) as well. We should again stress the important fact that (4.1) contains the lapse function that depends on $\mathbf{x}$ as well. Note that we only demand that this Hamiltonian annihilates the vacuum state functional while its action on other states of the theory is not specified. This is different from the standard constraint of general relativity where the Dirac analysis implies that *all* wave functionals should be annihilated by Hamiltonian and diffeomorphism constraints. On the other hand we will see below that the correct Hamiltonian treatment of the theory specified by the Hamiltonian above will lead to the requirement that all states should be annihilated by (4.1).

As usual we are interested in the Lagrangian formulation of given theory. In order to find it we consider the classical form of the Hamiltonian (4.1) and we also restrict ourselves to the following form of the Hamiltonian density $\mathcal{H}_0$

$$\mathcal{H}_0 = \kappa^2\sqrt{g}\left(\sqrt{1+\beta(-i\pi^{ij}+\sqrt{g}E^{ij})\frac{1}{g}\mathcal{G}_{ijkl}\left(i\pi^{kl}+\sqrt{g}\frac{1}{2}E^{kl}\right)}-1\right). \tag{4.4}$$

Then using (4.1) and (4.4) we find that the time derivative of g_{ij} is equal to

$$\partial_t g_{ij} = \{g_{ij}, H\} = \kappa^2\frac{N\beta}{\sqrt{g}}\frac{\mathcal{G}_{ijkl}\pi^{kl}}{\sqrt{1+\frac{\beta}{g}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl}+\beta E^{ij}\mathcal{G}_{ijkl}E^{kl}}}+\nabla_j N_i+\nabla_i N_j\,, \tag{4.5}$$

where we used the canonical Poisson brackets

$$\left\{g_{ij}(\mathbf{x}),\pi^{kl}(\mathbf{y})\right\} = \frac{1}{2}(\delta_i^k\delta_j^l+\delta_i^l\delta_j^k)\delta(\mathbf{x}-\mathbf{y}) \tag{4.6}$$

and the fact that

$$\begin{aligned}\left\{g_{ij}(\mathbf{x}),\int d^D\mathbf{y}N_k(\mathbf{y})\mathcal{H}^k(\mathbf{y})\right\} &= -2\int d^D\mathbf{y}N_k(\mathbf{y})\nabla_l(\left\{g_{ij}(\mathbf{x}),\pi^{kl}(\mathbf{y})\right\}) = \\ &= \nabla_j N_i(\mathbf{x})+\nabla_i N_j(\mathbf{x}).\end{aligned} \tag{4.7}$$

The equation (4.5) implies that it is natural to introduce the tensor $K_{ij} = \frac{1}{2N}(\partial_t g_{ij} - \nabla_j N_i - \nabla_i N_j)$ so that (4.5) can be written as

$$2K_{ij} = \kappa^2\frac{\beta}{\sqrt{g}}\frac{\mathcal{G}_{ijkl}\pi^{kl}}{\sqrt{1+\frac{\beta}{g}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl}+\beta E^{ij}\mathcal{G}_{ijkl}E^{kl}}}. \tag{4.8}$$

Clearly using this relation we can express π^{ij} as a function of g_{ij}, K_{ij}. Then after some algebra we find the Lagrangian in the form

$$\begin{aligned}L &= \int d^D\mathbf{x}(\partial_t g_{ij}\pi^{ij} - N\mathcal{H}_0 - N_i\mathcal{H}^i) = \\ &= -\kappa^2\int d^D\mathbf{x}\sqrt{g}N\left(\sqrt{1+\beta E^{ij}\mathcal{G}_{ijkl}E^{kl}}\sqrt{1-\frac{4}{\kappa^4\beta}K_{ij}\mathcal{G}^{ijkl}K_{kl}}-1\right).\end{aligned} \tag{4.9}$$

We see that this Lagrangian takes completely the same form as the Lagrangian given in (3.25). However it is crucial that in the new formulation the field N depends on $\mathbf{x}$ and t as well. In other words we derived the Hořava-Lifshitz $f(R)$ gravity theory without projectability condition.

4.1 Hamiltonian formalism

We see that the Lagrangian density (4.9) depends on $N(t,\mathbf{x})$ and $N^i(t,\mathbf{x})$ that can be interpreted as additional fields in the theory. Then when we proceed to the Hamiltonian formalism we find that the phase space of the theory is spanned by N, N^i with conjugate momenta π_N, π_i and metric components g_{ij} with conjugate momenta π^{ij}. The fact that the

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Lagrangian (4.9) does not contain time derivatives of N and N^i implies that the momenta $\pi^i(\mathbf{x}), \pi_N(\mathbf{x})$ vanish and form the primary constraints of the theory. Finally the standard analysis of constraints system implies that the Hamiltonian (4.1) with $\mathcal{H}_0$ given in (4.4) is a sum of the local constraints

$$\mathcal{H}_0(\mathbf{x}) \approx 0\,, \mathcal{H}^i(\mathbf{x}) \approx 0\ . \tag{4.10}$$

The quantum mechanical analogue of these constraints is the requirement that *all wave functionals* should be annihilated by them. Observe that this is more stronger requirement then the formulation of the local balance condition given in the first paragraph of this section. In summary, the consistency of the theory defined by (4.9) implies that at the classical level the Hamiltonian (4.1) should be sum of local constraints. The quantum mechanical formulation is that all wave functionals should be annihilated by the quantum Hamiltonian (4.1) again with $\hat{\mathcal{H}}_0$ given in (4.4).

Now we start to study the algebra of constraints $\mathcal{H}^i, \mathcal{H}_0$ when we presume the most general form of the constraint $\mathcal{H}_0$

$$\begin{aligned}\mathcal{H}_0 &= \kappa^2\sqrt{g}\left(\sum_{n=0}^{\infty} c_n(g_{ij})\left(Q^{\dagger ij}\frac{1}{g}\mathcal{G}_{ijkl}Q^{kl}\right)^n - c_0(g_{ij})\right) = \\ &= \kappa^2\sqrt{g}\sum_{n=1}^{\infty} c_n(g_{ij})\left(Q^{\dagger ij}\frac{1}{g}\mathcal{G}_{ijkl}Q^{kl}\right)^n .\end{aligned} \tag{4.11}$$

If we introduce the smeared form of the diffeomorphism constraint $\mathbf{T}_S(\zeta) = \int d^D\mathbf{x}\zeta_i(\mathbf{x})\mathcal{H}^i(\mathbf{x})$ we can easily determine Poisson brackets

$$\begin{aligned}\{\mathbf{T}_S(\zeta), g_{ij}\} &= -\zeta^k\partial_k g_{ij} - g_{jk}\partial_i\zeta^k - g_{ik}\partial_j\zeta^k\,, \\ \{\mathbf{T}(\zeta), \pi^{ij}\} &= -\partial_k(\pi^{ij}\zeta^k) + \pi^{jk}\partial_k\zeta^i + \pi^{ik}\partial_k\zeta^j\,, \\ \{\mathbf{T}_S(\zeta), \sqrt{g}\} &= -\zeta^k\partial_k\sqrt{g} - \partial_k\zeta^k\sqrt{g}\,, \\ \left\{\mathbf{T}_S(\zeta), \frac{1}{2}\frac{\delta W}{\delta g^{ij}}\right\} &= -\partial_k\left(\frac{1}{2}\zeta^k\frac{\delta W}{\delta g_{ij}}\right) + \frac{1}{2}\frac{\delta W}{\delta g_{ik}}\partial_j\zeta^k + \partial_i\zeta^k\frac{1}{2}\frac{\delta W}{\delta g_{kj}}.\end{aligned} \tag{4.12}$$

Then it is easy to find that

$$\begin{aligned}\{\mathbf{T}_S(\zeta), Q^{ij}\} &= -\partial_k(Q^{ij}\zeta^k) + \partial_k\zeta^i Q^{kj} + Q^{ik}\partial_k\zeta^j\,, \\ \left\{\mathbf{T}_S(\zeta), Q^{\dagger ij}\right\} &= -\partial_k\left(\zeta^k Q^{\dagger ij}\right) + \partial_k\zeta^i Q^{\dagger kj} + Q^{\dagger ik}\partial_k\zeta^j.\end{aligned} \tag{4.13}$$

For further purposes we also determine following Poisson bracket

$$\begin{aligned}\left\{\mathbf{T}_S(\zeta), \frac{1}{g}\mathcal{G}_{ijkl}\right\} &= (2\partial_k\zeta^k(g) + \zeta^k\partial_k(g))\frac{1}{g^2}\mathcal{G}_{ijkl} - \\ &-\frac{1}{g}(\partial_p\mathcal{G}_{ijkl}\zeta^p + \partial_i\zeta^p\mathcal{G}_{pjkl} + \partial_j\zeta^p\mathcal{G}_{ipkl} + \mathcal{G}_{ijpl}\partial_k\zeta^p + \mathcal{G}_{ijkp}\partial_j\zeta^p).\end{aligned} \tag{4.14}$$

Then it is easy to see

$$\left\{\mathbf{T}_S(\zeta), Q^{\dagger ij}\frac{1}{g}\mathcal{G}_{ijkl}Q^{kl}\right\} = -\partial_m\left(Q^{\dagger ij}\frac{1}{g}\mathcal{G}_{ijkl}Q^{kl}\right)\zeta^m. \tag{4.15}$$

Using this result and also the third equation in (4.12) we find

$$\begin{aligned}\{\mathbf{T}_S(\zeta),\mathcal{H}_0\} &= \kappa^2[-\zeta^k\partial_k\sqrt{g}-\partial_k\zeta^k\sqrt{g}]\sum_{n=1}^{\infty}c_n\left(Q^{\dagger ij}\frac{1}{g}\mathcal{G}_{ijkl}Q^{kl}\right)^n- \\ &\quad -\kappa^2\sqrt{g}\sum_{n=1}^{\infty}c_n\partial_m\left(Q^{\dagger ij}\frac{1}{g}\mathcal{G}_{ijkl}Q^{kl}\right)\zeta^m\left(Q^{\dagger ij}\frac{1}{g}\mathcal{G}_{ijkl}Q^{kl}\right)^{n-1}= \\ &= -\partial_k\zeta^k\mathcal{H}_0-\zeta^k\partial_k\mathcal{H}_0 \end{aligned} \tag{4.16}$$

and when we introduce the smeared form of the constraint $\mathcal{H}_0$

$$\mathbf{T}_T(f)=\int d^D\mathbf{x}f(\mathbf{x})\mathcal{H}_0(t,\mathbf{x}) \tag{4.17}$$

we obtain

$$\begin{aligned}\{\mathbf{T}_S(\zeta),\mathbf{T}_T(f)\} &= -\int d^D\mathbf{x}f(\mathbf{x})(\partial_k\zeta^k\mathcal{H}_0(\mathbf{x})+\zeta^k\partial_k\mathcal{H}_0(\mathbf{x}))= \\ &= \int d^D\mathbf{x}\partial_k f(\mathbf{x})\zeta^k\mathcal{H}_0=\mathbf{T}_T(\partial_k f\zeta^k). \end{aligned} \tag{4.18}$$

Finally the Poisson brackets of the diffeomorphism constraints is equal to

$$\{\mathbf{T}_S(\zeta),\mathbf{T}_S(\xi)\}=\mathbf{T}_S(\zeta^i\partial_i\xi-\xi^i\partial_i\zeta). \tag{4.19}$$

Now we come to the analysis of the most intricate Poisson bracket $\{\mathbf{T}_T(f),\mathbf{T}_T(\zeta)\}$. Note that the previous Poisson brackets were valid for any form of the constraint $\mathcal{H}_0$. On the other hand we can certainly find an equivalent constraint using following observation. The Hamiltonian constraint has the form $\mathcal{H}_0=f(Q^{\dagger ij}\mathcal{G}_{ijkl}Q^{kl})$. Then instead imposing the constraint $\mathcal{H}_0\approx 0$ we can impose the constraint $\sqrt{g}Q^{\dagger ij}\mathcal{G}_{ijkl}Q^{kl}\approx 0$. This fact simplifies the analysis considerably however it is still very intricate as was shown for example in [37] where the analysis of the constraint algebra of 3+1 dimensional Hořava-Lifshitz theory was performed with the result that the Poisson bracket of the constraint $\sqrt{g}Q^{\dagger ij}\mathcal{G}_{ijkl}Q^{kl}\approx 0$ is not closed but it generates new additional ones. The upshot of this analysis is that it seems that the resulting theory does not contain any physical degrees of freedom. This seems to be a serious problem of the Hořava-Lifshitz theory without projectability condition. On the other hand there exists an alternative procedure how to solve the constraint $\mathcal{H}_0\approx 0$. This idea was suggested in the original Hořava work [3]. Explicitly, the form of the constraint $\mathcal{H}_0(\mathbf{x})\approx 0$ suggests that the constraint $\mathcal{H}_0(\mathbf{x})\approx 0$ can be solved by collection of constraints $Q^{ij}(\mathbf{x})\approx 0$. In other words we propose following alternative set of constraints of $f(R)$ Hořava-Lifshitz gravity

$$\mathcal{H}^i(\mathbf{x})\approx 0\,,\quad Q^{ij}(\mathbf{x})\approx 0 \tag{4.20}$$

or their smeared form

$$\mathbf{T}_S(\zeta)=\int d^D\mathbf{x}\zeta_i(\mathbf{x})\mathcal{H}^i(\mathbf{x})\,,\quad \mathbf{Q}(\Lambda)=\int d^D\mathbf{x}\Lambda_{ij}(\mathbf{x})Q^{ij}(\mathbf{x})\,. \tag{4.21}$$

Let us now show that this set of constraints forms the closed algebra. Since

$$\left\{Q^{ij}(\mathbf{x}),Q^{kl}(\mathbf{y})\right\}=-\frac{i}{2}\frac{\delta}{\delta g^{ij}(\mathbf{x})}\frac{\delta W}{\delta g^{kl}(\mathbf{y})}+\frac{i}{2}\frac{\delta}{\delta g^{kl}(\mathbf{y})}\frac{\delta W}{\delta g^{ij}(\mathbf{x})}=0 \tag{4.22}$$

we easily find that

$$\{\mathbf{Q}(\Lambda), \mathbf{Q}(\Gamma)\} = 0 \ . \tag{4.23}$$

Further, using (4.13) we find

$$\begin{aligned}\{\mathbf{T}_S(\zeta), \mathbf{Q}(\Lambda)\} &= \int d^D\mathbf{x}(\partial_k\Lambda_{ij}Q^{ij}\zeta^k + \partial_k\zeta^i Q^{kj} + Q^{ik}\partial_k\zeta^j) = \\ &= \mathbf{Q}(\partial_k\Lambda_{ij}\zeta^k + \partial_i\zeta^k\Lambda_{kj} + \Lambda_{ik}\partial_j\zeta^k) \ .\end{aligned} \tag{4.24}$$

These Poisson brackets together with (4.19) imply that the algebra of the constraints (4.21) is closed. On the other hand as was stressed originally in [3] this set of constraints is certainly too strong and it turns out that the resulting theory is topological without any local excitations. This conclusion however suggests that the Hořava-Lifshitz theory of gravity without projectability condition has natural physical interpretation as the topological theory of gravity.

A Ultralocal gravity

In this appendix we present an example of the Hořava-Lifshitz $f(R)$ gravity that has closed algebra of constraints. Using terminology introduced in [2] we call this theory as *ultralocal Hořava-Lifshitz $f(R)$ gravity*.

The simplest example of the ultralocal theory is characterized by condition that

$$E^{ij} = 0 \ . \tag{A.1}$$

Since in this case $Q^{ij} = -Q^{\dagger ij}$ we find

$$\begin{aligned}\left\{Q^{\dagger ij}(\mathbf{x}), Q^{kl}(\mathbf{y})\right\} &= 0\,, \\ \left\{Q^{ij}(\mathbf{x}), \frac{1}{g}\mathcal{G}_{klmn}(\mathbf{y})\right\} &= -\left\{Q^{\dagger ij}(\mathbf{x}), \frac{1}{g}\mathcal{G}_{klmn}(\mathbf{y})\right\}\end{aligned} \tag{A.2}$$

and consequently

$$\left\{Q^{\dagger ij}\frac{1}{g}\mathcal{G}_{ijkl}Q^{kl}(\mathbf{x}), Q^{\dagger mn}\frac{1}{g}\mathcal{G}_{mnpq}Q^{pq}(\mathbf{y})\right\} = 0 \ . \tag{A.3}$$

Then it is easy to determine the Poisson brackets of the constraints $\mathcal{H}_0 \approx 0$ using the fact that

$$\begin{aligned}\{\mathcal{H}_0(\mathbf{x}), \mathcal{H}_0(\mathbf{y})\} &= \int d\mathbf{x}'d\mathbf{y}'\frac{\delta\mathcal{H}_0(\mathbf{x})}{\delta(Q^{\dagger ij}\mathcal{G}_{ijkl}Q^{kl})(\mathbf{x}')}\left\{(Q^{\dagger ij}\mathcal{G}_{ijkl}Q^{kl})(\mathbf{x}'), (Q^{\dagger ij}\mathcal{G}_{ijkl}Q^{kl})(\mathbf{y}')\right\}\times \\ &\quad\times\frac{\delta\mathcal{H}_0(\mathbf{y})}{\delta(Q^{\dagger ij}\mathcal{G}_{ijkl}Q^{kl})(\mathbf{y}')} = 0.\end{aligned} \tag{A.4}$$

Let us now consider the second example of ultralocal theory when W has the form

$$W = \Lambda\sqrt{g} \ . \tag{A.5}$$

In this case we easily find

$$Q^{\dagger ij} = -i\pi^{ij} + \frac{1}{4}\Lambda g^{ij}\sqrt{g}\,, \quad Q^{ij} = i\pi^{ij} + \frac{1}{4}\Lambda g^{ij}\sqrt{g}\,. \tag{A.6}$$

Now the Poisson brackets between $Q^{\dagger ij}$ and Q^{kl} is non-zero

$$\left\{Q^{\dagger ij}(\mathbf{x}), Q^{kl}(\mathbf{y})\right\} = -i\frac{\Lambda}{4}(g^{ik}g^{jl} + g^{il}g^{kl})\sqrt{g}\delta(\mathbf{x}-\mathbf{y}). \tag{A.7}$$

It is important that this Poisson bracket is proportional to $\delta(\mathbf{x}-\mathbf{y})$ and does not contain derivative of delta function. Then with the help of this result and the second equation in (A.2) we again find that

$$\left\{Q^{\dagger ij}\frac{1}{g}\mathcal{G}_{ijkl}Q^{kl}(\mathbf{x}), Q^{\dagger ij}\frac{1}{g}\mathcal{G}_{ijkl}Q^{kl}(\mathbf{y})\right\} = 0 \tag{A.8}$$

and as follows from (A.4) the Poisson brackets of the Hamiltonian constraints vanish.

In summary, the ultralocal $f(R)$ Hořava-Lifshitz gravity has the same nice property as the ultralocal theory of gravity [67].

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